On a boundary layer phenomenon in acoustic media

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WONAPDE 2016, Concepción

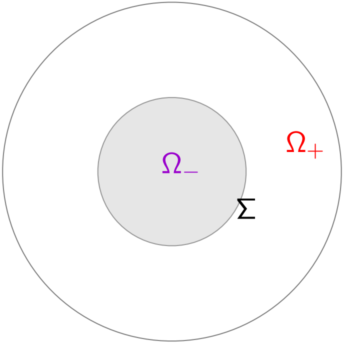
January 11-16, 2016

Introduction

A 3-D Problem set in heterogeneous acoustic media

We solve the Helmholtz equation (with a Dirichlet B.C.) :

$$-\operatorname{div}\left(\frac{1}{\rho}\nabla \mathrm{u}\right) - \frac{\omega^2}{\rho c^2}\mathrm{u} = f \quad \text{in} \quad \Omega$$



- $\Omega_- = \{r < r_t\}$ Acoustic medium : Density ρ^-, Velocity c^-
- $\Sigma = \{r = r_t\}$: Interface
- $\Omega_+ = \Omega \setminus \overline{\Omega_-}$ Acoustic medium : Density ρ^+, Velocity c^+

Framework

Data and right-hand side

Assumption

(i) *The density* ρ *and the velocity c are* **smooth** *and strictly* **positive** *functions*

(ii) *The density* ρ^+ *satisfies*

$$\rho^+(r) = \rho_S(r_t)\exp\left(-\alpha(r - r_t)\right)$$

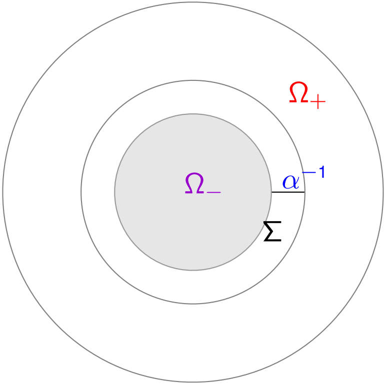
where the parameter α *is large.*

Assumption

The right-hand side f has a support in Ω_-

Aim

A boundary layer phenomenon occurs : rapid decay of acoustic fields (pressure and velocity) inside the medium Ω_+

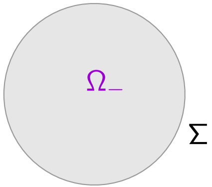


Goal : *(i)* Describing the boundary layer phenomenon

(ii) Deriving and solving approximate models

Approach : an Asymptotic Method

1. Describing the boundary layer phenomenon with a multiscale expansion in power series of α^{-1}
2. Deriving Equivalent Boundary Conditions on Σ to replace the boundary layer inside Ω_+
3. Applying a Finite Element Method to solve the acoustic equation in Ω_- with an Equivalent Boundary Condition



Related references on skin effect in electromagnetism

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Outline

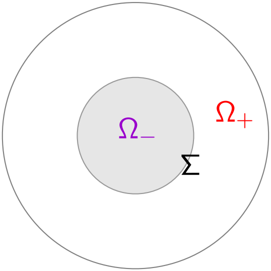
Outline

Problem (P_α)

Helmholtz equation :

$$-\operatorname{div}\left(\frac{1}{\rho}\nabla u_\alpha\right) - \frac{\omega^2}{\rho c^2} u_\alpha = f \quad \text{in} \quad \Omega$$

with an homogeneous Dirichlet boundary condition on $\partial\Omega$.



Here $\rho^+(r) = \rho_S(r_t)\exp\left(-\alpha(r - r_t)\right)$ and $\alpha \gg 1$

Problem $(\mathbf{P}_\alpha)$

Issue : *Uniform* H^1 *estimates* for solutions u_α of $(\mathbf{P}_\alpha)$ as $\alpha \to \infty$?

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Assumption (SA)

The angular frequency ω is not an eigenfrequency of the problem

$$\begin{cases} \operatorname{div}\left(\dfrac{1}{\rho}\nabla u^-\right) + \dfrac{\omega^2}{\rho c^2} u^- = 0 & \textit{in} \quad \Omega_- \\ u^- = 0 & \textit{on} \quad \Sigma \end{cases}$$

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Theorem

Assume Ω_+ is a **dissipative medium**. *Then under Assumption (SA), there exists $\alpha_0 > 0$ s.t. for all $\alpha \in (\alpha_0, \infty)$, the problem $(\mathbf{P}_\alpha)$ with a right-hand side $f \in \mathrm{L}^2(\Omega)$ has a unique solution $u_\alpha \in \mathrm{H}^1_0(\Omega)$, and*

$$\|u_\alpha^-\|_{1,\Omega_-} + \alpha\|u_\alpha^+\|_{1,\Omega_+} \leqslant C\|f\|_{0,\Omega}$$

Application: Convergence of asymptotic expansion for large parameter α

Problem (P_δ)

Small parameter δ :

$$\delta = \frac{1}{\alpha} \longrightarrow 0 \quad \text{when} \quad \alpha \to \infty$$

Problem $(\mathbf{P}_\delta)$

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$$\delta = \frac{1}{\alpha} \longrightarrow 0 \quad \text{when} \quad \alpha \to \infty$$

Problem $(\mathbf{P}_\delta)$ writes :

$$\begin{aligned}
&-\operatorname{div}\left(\frac{1}{\rho}\nabla \mathrm{u}_\delta^-\right) - \frac{\omega^2}{\rho c^2}\mathrm{u}_\delta^- = f && \text{in} \quad \Omega_- \\
&-\Delta \mathrm{u}_\delta^+ - \frac{1}{\delta}\partial_r \mathrm{u}_\delta^+ - \frac{\omega^2}{c^2}\mathrm{u}_\delta^+ = 0 && \text{in} \quad \Omega_+ \\
&\mathrm{u}_\delta^+ = \mathrm{u}_\delta^- && \text{on} \quad \Sigma \\
&\partial_{\mathbf{n}} \mathrm{u}_\delta^+ = \partial_{\mathbf{n}} \mathrm{u}_\delta^- && \text{on} \quad \Sigma \\
&\mathrm{u}_\delta^+ = 0 && \text{on} \quad \partial\Omega
\end{aligned}$$

Outline

Asymptotic Expansion

Overview

Deriving an Asymptotic Expansion for the solution $\left(u_\delta^-, u_\delta^+\right)$ of $\left(\mathbf{P}_\delta\right)$ when $\delta \to 0$:

$$u_\delta^-(\mathbf{x}) = u_0^-(\mathbf{x}) + \delta u_1^-(\mathbf{x}) + \delta^2 u_2^-(\mathbf{x}) + \delta^3 u_3^-(\mathbf{x}) + \cdots \quad \text{in} \quad \Omega_-$$

$$u_\delta^+(\mathbf{x}) = u_0^+(\mathbf{x};\delta) + \delta u_1^+(\mathbf{x};\delta) + \delta^2 u_2^+(\mathbf{x};\delta) + \delta^3 u_3^+(\mathbf{x};\delta) + \cdots \quad \text{in} \quad \Omega_+ ,$$

$$\text{with} \quad u_j^+(\mathbf{x};\delta) = \chi(r)\,\mathfrak{U}_j\left(\theta, \phi, \frac{r - r_t}{\delta}\right)$$

Method based on the Scaling :

$$S = \frac{r - r_t}{\delta}$$

Equations for the terms $\mathfrak{U}_n$ and u_n^-

$$\begin{cases} -\partial_S^2 \mathfrak{U}_n - \partial_S \mathfrak{U}_n & = & \displaystyle\sum_{p=1}^{n} \mathrm{A}_p \mathfrak{U}_{n-p} & \text{in} \quad \Sigma \times (0, +\infty)\,, \\ \partial_S \mathfrak{U}_n & = & \partial_{\mathbf{n}} \mathrm{u}_{n-1}^- & \text{on} \quad \Sigma\,, \end{cases}$$

and

$$\begin{cases} -\operatorname{div}\left(\frac{1}{\rho}\nabla \mathrm{u}_n^-\right) - \frac{\omega^2}{\rho c^2}\mathrm{u}_n^- & = & f\delta_n^0 & \text{in} \quad \Omega_-\,, \\ \mathrm{u}_n^- & = & \mathfrak{U}_n & \text{on} \quad \Sigma\,. \end{cases}$$

First terms

1. $$\mathfrak{U}_0 = 0$$

2. $$\begin{cases} -\operatorname{div}\left(\frac{1}{\rho}\nabla \mathrm{u}_0^-\right) - \frac{\omega^2}{\rho c^2}\mathrm{u}_0^- = f & \text{in} \quad \Omega_- \\ \mathrm{u}_0^- = 0 & \text{on} \quad \Sigma \end{cases}$$

3. $$\mathfrak{U}_1(\cdot, S) = -\partial_{\mathbf{n}} \mathrm{u}_0^- \, \mathrm{e}^{-S}, \qquad S \in (0, \infty)$$

4. $$\begin{cases} -\operatorname{div}\left(\frac{1}{\rho}\nabla \mathrm{u}_1^-\right) - \frac{\omega^2}{\rho c^2}\mathrm{u}_1^- = 0 & \text{in} \quad \Omega_- \\ \mathrm{u}_1^- = -\partial_{\mathbf{n}} \mathrm{u}_0^- & \text{on} \quad \Sigma \end{cases}$$

Next terms

- $$\mathfrak{U}_2(\cdot, S) = (a_2 + b_2 S)\,\mathrm{e}^{-S}, \qquad S \in (0, \infty)$$

  Here
  $$\begin{cases} a_2 = -\partial_{\mathbf{n}} \mathrm{u}_1^- + \dfrac{2}{r_t} \partial_{\mathbf{n}} \mathrm{u}_0^- , \\ b_2 = \dfrac{2}{r_t} \partial_{\mathbf{n}} \mathrm{u}_0^- . \end{cases}$$

- $$\begin{cases} -\operatorname{div}\left(\frac{1}{\rho} \nabla \mathrm{u}_2^-\right) - \frac{\omega^2}{\rho c^2} \mathrm{u}_2^- = 0 & \text{in} \quad \Omega_- , \\ \mathrm{u}_2^- = -\partial_{\mathbf{n}} \mathrm{u}_1^- + \dfrac{2}{r_t} \partial_{\mathbf{n}} \mathrm{u}_0^- & \text{on} \quad \Sigma . \end{cases}$$

Next terms

- $$\mathfrak{U}_3(\theta,\phi,S) = (a_3(\theta,\phi) + S b_3(\theta,\phi) + S^2 c_3(\theta,\phi))\mathrm{e}^{-S}, \quad S \in (0,\infty)$$

  Here

  $$\begin{cases} a_3 = -\partial_{\mathbf{n}} \mathrm{u}_2^- + \dfrac{2}{r_t}\partial_{\mathbf{n}} \mathrm{u}_1^- - \left\{ \frac{6}{r_t^2} + \frac{\omega^2}{c^2} + \Delta_\Sigma \right\} \partial_{\mathbf{n}} \mathrm{u}_0^- , \\ b_3 = \dfrac{2}{r_t}\partial_{\mathbf{n}} \mathrm{u}_1^- - \left\{ \frac{6}{r_t^2} + \frac{\omega^2}{c^2} + \Delta_\Sigma \right\} \partial_{\mathbf{n}} \mathrm{u}_0^- , \\ c_3 = -\dfrac{3}{r_t^2}\partial_{\mathbf{n}} \mathrm{u}_0^- . \end{cases}$$

- $$\begin{cases} -\operatorname{div}\left(\frac{1}{\rho}\nabla \mathrm{u}_3^-\right) - \frac{\omega^2}{\rho c^2}\mathrm{u}_3^- = 0 & \text{in} \quad \Omega_- , \\ \mathrm{u}_3^- = -\partial_{\mathbf{n}} \mathrm{u}_2^- + \dfrac{2}{r_t}\partial_{\mathbf{n}} \mathrm{u}_1^- - \left\{ \frac{6}{r_t^2} + \frac{\omega^2}{c^2} + \Delta_\Sigma \right\} \partial_{\mathbf{n}} \mathrm{u}_0^- & \text{on} \quad \Sigma . \end{cases}$$

Existence and regularity of the asymptotics

Proposition

Let $k \in \mathbb{N}$. Assume that $f \in \mathrm{H}^k(\Omega_-)$ and $\rho \in \mathcal{C}^\infty(\overline{\Omega_-})$. Then it is possible to derive the first $(k+2)$ terms $(\mathsf{u}_j^-, \mathfrak{U}_j)$, and

$$\mathsf{u}_0^- \in \mathrm{H}^{k+2}(\Omega_-), \mathsf{u}_1^- \in \mathrm{H}^{k+1}(\Omega_-), \cdots, \mathsf{u}_{k+1}^- \in \mathrm{H}^1(\Omega_-)$$

and

$$\mathfrak{U}_1 \in \mathrm{H}^{k+\frac{1}{2}}(\Sigma \times \mathbb{R}^+), \mathfrak{U}_2 \in \mathrm{H}^{k-\frac{1}{2}}(\Sigma \times \mathbb{R}^+), \cdots, \mathfrak{U}_{k+1} \in \mathrm{H}^{\frac{1}{2}}(\Sigma \times \mathbb{R}^+)$$

Outline

Equivalent Conditions on Σ

Overview

We identify a simpler problem satisfied by

$$\mathrm{u}_{k,\delta}^{-} := \mathrm{u}_0^{-} + \delta \mathrm{u}_1^{-} + \delta^2 \mathrm{u}_2^{-} + \cdots + \delta^k \mathrm{u}_k^{-} \quad \text{up to} \quad \mathcal{O}(\delta^{k+1})$$

The simpler problem *of order* $k+1$ writes

$$(\mathbf{P}_\delta^k) \quad \begin{cases} -\operatorname{div}\left(\frac{1}{\rho}\nabla \mathrm{u}_k^\delta\right) - \frac{\omega^2}{\rho c^2}\mathrm{u}_k^\delta & = \quad f \quad \text{in} \quad \Omega_- \,, \\ \mathrm{u}_k^\delta + \mathrm{D}_{k,\delta}(\partial_{\mathbf{n}} \mathrm{u}_k^\delta) & = \quad 0 \quad \text{on} \quad \Sigma \,, \end{cases}$$

with $\mathrm{D}_{k,\delta}$ a surfacic differential operator

Equivalent Conditions

1. Order 1 :

$$\mathrm{u}_0 = 0 \quad \text{on} \quad \Sigma$$

2. Order 2 :

$$\mathrm{u}_1^\delta + \delta \partial_{\mathbf{n}} \mathrm{u}_1^\delta = 0 \quad \text{on} \quad \Sigma$$

3. Order 3 :

$$\mathrm{u}_2^\delta + \delta \left(1 - \frac{2\delta}{r_t} \right) \partial_{\mathbf{n}} \mathrm{u}_2^\delta = 0 \quad \text{on} \quad \Sigma$$

4. Order 4 :

$$\mathrm{u}_3^\delta + \delta \left(1 - \frac{2\delta}{r_t} + \delta^2 \left\{ \frac{6}{r_t^2} - \frac{\omega^2}{c^2} - \Delta_\Sigma \right\} \right) \partial_{\mathbf{n}} \mathrm{u}_3^\delta = 0 \quad \text{on} \quad \Sigma$$

Dirichlet-to-Neumann Conditions

- Order 4 :

$$\delta^{-1}\left(\left(1+\frac{2\delta}{r_t}\right)\mathbb{I}-\delta^2\left\{\frac{2}{r_t^2}+\frac{\omega^2}{c^2}+\Delta_\Sigma\right\}\right)\mathrm{v}_3^\delta+\partial_{\mathbf{n}}\mathrm{v}_3^\delta=0 \quad \text{on} \quad \Sigma$$

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For all $k \in \{1,2,3\}$, the simpler problem writes

$$(\mathbf{Q}_\delta^k) \quad \left\{\begin{array}{rcll} -\operatorname{div}(\frac{1}{\rho}\nabla \mathrm{v}_k^\delta)-\frac{\omega^2}{\rho c^2}\mathrm{v}_k^\delta & = & f & \text{in} \quad \Omega_- \,, \\ \mathrm{N}_{k,\delta}(\mathrm{v}_k^\delta)+\partial_{\mathbf{n}}\mathrm{v}_k^\delta & = & 0 & \text{on} \quad \Sigma \,, \end{array}\right.$$

with $\mathrm{N}_{k,\delta}$ a surfacic differential operator.

Stability and convergence results

Theorem

Under Assumption (SA), there exists $\delta_0 > 0$ s.t. for all $\delta \in (0, \delta_0)$, the problem $(\mathbf{Q}_\delta^k)$ with data $f \in \mathrm{L}^2(\Omega_-)$ has a unique solution $\mathrm{v}_k^\delta \in \mathrm{V}_k$, and

$$\|\mathrm{v}_k^\delta\|_{1,\Omega_-} \leqslant C\|f\|_{0,\Omega_-}$$

$$\|\mathrm{u}_\delta - \mathrm{v}_\delta^k\|_{1,\Omega_-} \leqslant C_k \delta^{k+1}$$

Functional setting : $\mathrm{v}_k^\delta \in \mathrm{V}_k$ where

$$\mathrm{V}_k = \mathrm{H}^1(\Omega_-) \quad \text{when} \quad k = 1, 2$$

and

$$\mathrm{V}_3 = \{\mathrm{u} \in \mathrm{H}^1(\Omega_-) \mid \ \mathrm{u}|_\Sigma \in \mathrm{H}^1(\Sigma)\}$$

Proof of the stability result

Lemma

Under Assumption (SA), there exists $\delta_0 > 0$ s.t. for all $\delta \in (0, \delta_0)$, any solution $\mathsf{v}_k^\delta \in \mathrm{V}_k$ of problem $(\mathbf{Q}_\delta^k)$ with a data $f \in \mathrm{L}^2(\Omega_-)$ satisfies

$$\|\mathsf{v}_k^\delta\|_{0,\Omega_-} \leqslant C\|f\|_{0,\Omega_-} .$$

Sketch of the proof of the Lemma

Assume : there exists $(\mathrm{u}_m) \in (\mathrm{V}_k)^{\mathbb{N}}$ satisfying $(\mathbf{Q}^k_{\delta_m})$ with $\delta_m \to 0$ and $f_m \in \mathrm{L}^2(\Omega_-)$ such that

$$\|\mathrm{u}_m\|_{0,\Omega_-} = 1 \quad \text{and} \quad \|f_m\|_{0,\Omega_-} \to 0$$

We prove successively that :

1. $$\|\mathrm{u}_m\|_{1,\Omega_-} \leqslant C$$
2. there exists a subsequence (u_m) s.t.

   $$\mathrm{u}_m \to \mathrm{u} \quad \text{in} \quad \mathrm{L}^2(\Omega_-) \quad \text{and} \quad \nabla \mathrm{u}_m \rightharpoonup \nabla \mathrm{u} \quad \text{in} \quad \mathbf{L}^2(\Omega_-)$$

   and $\mathrm{u}_m \to 0 \quad \text{in} \quad \mathrm{L}^2(\Sigma)$
3. $$\|\mathrm{u}\|_{0,\Omega_-} = 1$$
4. Using Assumption (SA), we prove that $\mathrm{u} = 0$. Contradiction.

Outline

Numerical method

Helmholtz equation with radial data

Using a decomposition in spherical harmonics we solve a sequence of 1-D problems with a Finite Element Method.

- 1-D Computational domain :
  Approximate models : $\Omega_- = (0;1)$
  Reference solutions : $\Omega = (0; R_\Omega)$ such that $\rho^+(R_\Omega) = 10^{-15}$
- $\mathbb{P}_{10}$-finite elements (Lagrange) available in the Library Montjoie
- Data :
  1. Artificial data
  2. Realistic data

Framework

Artificial data

- The density is given as

$$\rho(r) = \begin{cases} D_1 & \text{if } r < 1 \\ D_1\, e^{-(r-1)/\delta} & \text{otherwise} \end{cases}$$

$$D_1 = 2 \times 10^{-4}\,\mathrm{kg}/\mathrm{m}^3,$$

- The velocity is constant

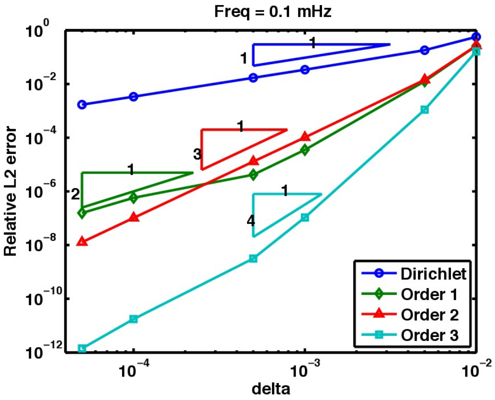
$$c = 8 \times 10^3\,\mathrm{m}/\mathrm{s}$$

- **Radial gaussian source** located at the radius $r = 0.8$
- $\omega = 2\pi f_0$ where $f_0 \in \{7 \times 10^4 Hz, 7 \times 10^5 Hz\}$

Convergence results

Freq. $f_0 = 7 \times 10^4$ **Hz**

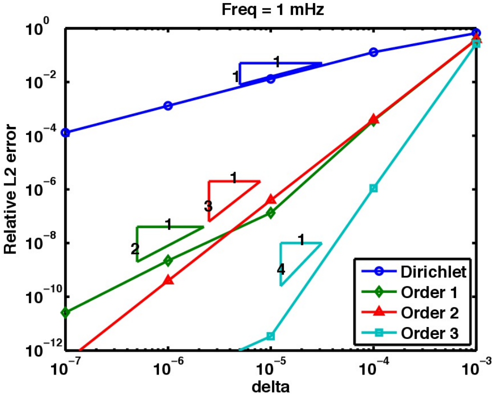
Relative L^2-error between exact solution and approximate solutions versus δ



Convergence results

Freq. $f_0 = 7 \times 10^5$ **Hz**

Relative L^2-error between exact solution and approximate solution versus δ



Realistic setting

Using the data given by the Standard Solar Model

CHRISTENSEN-DALSGAARD ET AL, THE CURRENT STATE OF SOLAR MODELING. SCIENCE, 1996

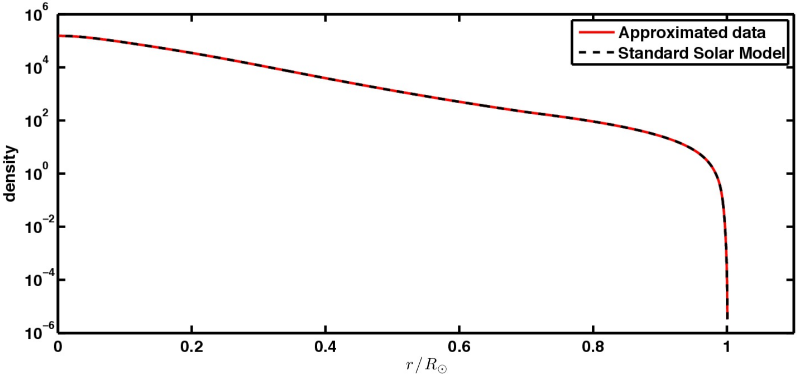
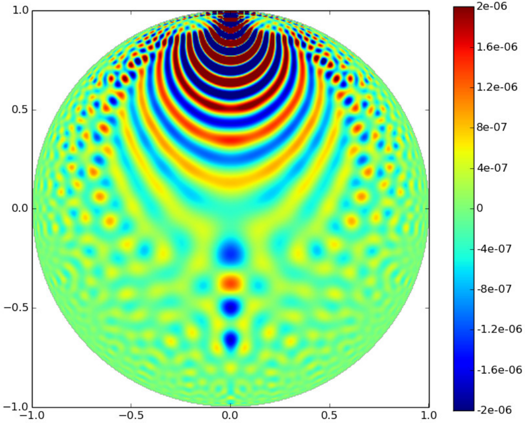


Figure : Exact value and approximated value of ρ^{sun} versus the relative radius $r/R_\odot$.

Realistic setting

Reference solution in the plane Oxz

Real part of $u/\sqrt{\rho}$ when $\delta = 1/7000$

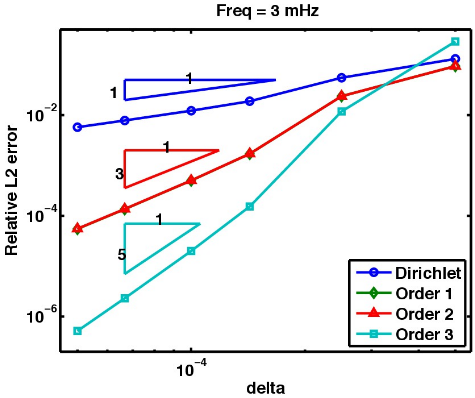


The source is a 3-D gaussian and we use 100 spherical harmonics

Realistic setting

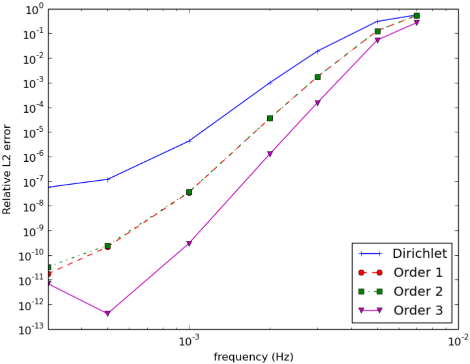
Convergence results

Relative L^2-error between reference solution and approximate solution versus δ



Realistic setting

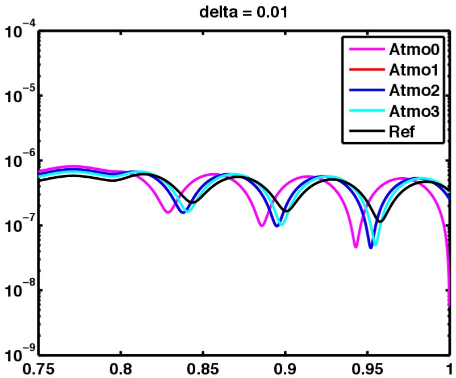
Comparison of equivalent boundary conditions for different frequencies from 0.3 mHz until 7 mHz.



Thank you for your attention

Numerical tests with artificial data

Qualitative Comparisons

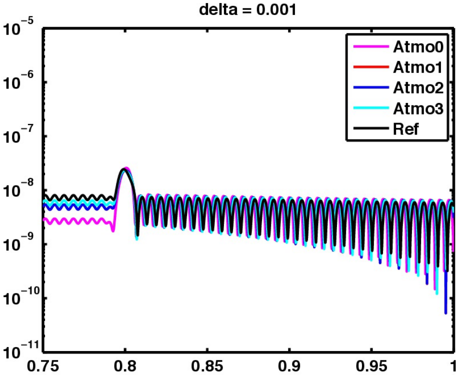


Reference solution and solutions using equivalent boundary conditions.

$f_0 = 7 \times 10^4$ Hz

Numerical tests with artificial data

Qualitative Comparisons



Reference solution and solutions using atmosphere boundary conditions.

$f_0 = 7 \times 10^5$ Hz

Realistic setting

- The density is given as

$$\rho_0(r) = \left(1 + \frac{2i\gamma}{\omega}\right) \begin{cases} \rho^{\text{sun}} & \text{if } r/R_\odot \leq 1.0007126 \\ D_1\, e^{-(r/R_\odot - 1)/\delta} & \text{otherwise} \end{cases}$$

$$D_1 \approx 3.292 \times 10^{-6}\,\text{kg}/\text{m}^3, \gamma = \frac{\omega}{100}$$

- The velocity is given as

$$c_0(r) = \begin{cases} c^{\text{sun}} & \text{if } r/R_\odot \leq 1.0007126 \\ C_1 & \text{otherwise} \end{cases}$$

$$C_1 \approx 6.865 \times 10^3\,\text{m}/\text{s}$$

- The source is a 3-D gaussian:

$$f(x) = \exp(-\beta||x - x_0||^2)$$

Here $x_0 \approx (0, 0, 0.981)$ and β is chosen such that $\exp(-0.015\,\beta) = 10^{-6}$.